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ON THE MANIFOLDS OF POSITIVE CURVATURE

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Abstract

In this paper invariant metrics on Lie group $G = S^3 \times R$ are studied and it is found lower and upper bounds for the sectional curvature's of the manifold $G = S^3 \times R$.

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1 Preliminaries

Let M be n-dimensional smooth Riemannian manifold with Riemannian metric g and $V(M)$ be a set of smooth vector fields on M. In article everywhere under the smoothness is understood the smoothness of class C^∞.

A Riemannian metric g induces a metric connection (Levi-Civita connection) ∇. Recall that, the connection is a mapping $\nabla : V(M) \times V(M) \to V(M)$, denoted by $(X, Y) \to \nabla_X Y$, and it has the following properties:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
3. $\nabla_{fX} Y = f\nabla_X Y$,
4. $\nabla_X fY = X(f)Y + f\nabla_X Y$, where f is a smooth function.

Lie bracket of vector fields X, Y we denote by $[X, Y]$. Levi-Civita connection ∇ and Lie bracket $[X, Y]$ are related to

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Definition 1. *The linear map $R : V(M) \times V(M) \times V(M) \to V(M)$, denoted by the formula*

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

is called the curvature tensor and the scalar value $k(X, Y) = \langle R_{XY}Y, X\rangle$ is called the Riemannian curvature [1].

Definition 2. *The sectional curvature of the Riemannian manifold M in the point p for the two-dimensional direction σ determined by the vectors $u, v \in T_pM$ is given by the rule*

$$K_{u,v} = \frac{k(u, v)}{|u|^2|v|^2 - \langle u, v\rangle^2} = \frac{\langle R_{uv}v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2}.$$

It is known that, if a manifold is a two-dimensional surface, immersed in a three-dimensional Euclidean space, then the sectional curvature coincides with the Gaussian curvature of the surface.

If the sectional curvature of $K_{u,v}$ is constant for all planes σ in T_pM and for every points $p \in M$, then the manifold M is called a manifold of constant curvature.

Recall that a Lie group is a group G that is in the time a smooth manifold, and its group actions and smooth structures are connected by the requirement that the maps $\varphi : G \to G$, $\psi : G \times G \to G$ defined by the equalities $\varphi(g) = g^{-l}$, $\psi(g, h) = gh$, maps φ and ψ were smooth.

The set of all real nonsingular $(n \times n)$ matrices with ordinary matrix multiplication forms a Lie group, denoted by $GL\,(n)$. Indeed, the set of all $(n \times n)$ matrices $A = (a^{ij})$, including singular ones, forms a vector space of dimension n^2. Its basis can serve as a matrix, one element of which is one, and the rest are zeros. Then $GL\,(n)$ is an open subset of R^{n^2} distinguished by the condition $\det A \neq 0$.

Definition 3. *Each element g of the Lie group G can be associated with its two automorphisms $L_g : G \to G$ and $R_g : G \to G$ acting by the rule $L_gh = gh,\ R_gh = hg$. These maps are called left and right translations.*

Definition 4. *A vector field X on the Lie group G is called left-invariant if for any $g \in G$ the equality holds $dL_gX = X$.*

For any vector $u \in T_eG$, where e is the unit element of G, $X_g = (d_eL_g)u$ defines a left-invariant vector field.

The Riemannian metric $\langle \cdot\, , \cdot \rangle$ on a Lie group G is called left-invariant, if for every $g, h \in G$ and for every tangent vectors $u, v \in T_gG$ it satisfies

$$\langle\, d\,L_h u, d\,L_h v\rangle_{hg} = \langle u,\, v\rangle_g. \tag{1}$$

The requirement (1) is equivalent to requiring that any left-invariant vector fields $\langle X, Y\rangle = \text{const}$.

The Riemannian metric, which is both left-and right-invariant, is called bi-invariant. Such metrics do not exist on all Lie groups. The following Theorem holds [4].

Theorem 1. *There exists at least one bi-invariant Riemannian metric on each compact Lie group.*

Further, we give properties of bi-invariant metrics in the form of theorems [3].

Theorem 2. *For a bi-invariant metric on the Lie group G and any left-invariant vector fields X, Y, Z on the Lie group G the following equality is true*

$$\langle X, [Y, Z]\rangle = \langle [X, Y]\,, Z\rangle\,.$$

Theorem 3. *For left-invariant vector fields X, Y the Levi-Civita connection of a bi-invariant Riemannian metric on the Lie group G has the form*

$$\nabla_X Y = \frac{1}{2}\,[X, Y]\,.$$

Volume 1, Issue 1 (2018)

Theorem 4. *For a bi-invariant metric on the Lie group G, the sectional curvature at the point $p \in G$ in the direction $\sigma = X_p \wedge Y_p$ is expressed by the formula*

$$K_\sigma (X_p \wedge Y_p) = \frac{1}{4} \frac{\langle [X, Y], [X, Y] \rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}\bigg|_p ,$$

where X, Y are left-invariant vector fields.

2 Main result

It is known that if $\varphi : M \to M$ is a differentiable mapping, then $[d\varphi X, d\varphi Y] = d\varphi[X, Y]$ [1].

Let the path $U(t) = (u^{ij}(t))$ exit from the unit E of the group $GL(n)$. We denote its initial velocity by $X_E = (x^{ij})$, where $x^{ij} = \frac{d}{dt} u^{ij}(t) \mid_{t=0}$.

We define the field X at the point $A = (a^{ij})$ as follows

$$X_A = dL_A(X_E) = \frac{d}{dt}\left(\sum_k a^{ik} u^{ki}(t)\right) = AX_E\,. \qquad (*)$$

It follows from the relation (*), that the Lie bracket has the form $[X, Y] = XY - YX$.

We now consider the group $SO(n)$ of real orthogonal $(n \times n)$ matrices with the condition $\det A \neq 0$. The sectional curvature of the Lie group $SO(n)$ is expressed as follows [3]:

$$K_\sigma = \frac{1}{4}\ \frac{\sum\limits_{i<j}\left(\sum\limits_k X^{ik}y^{kj} - \sum\limits_k y^{ik}X^{kj}\right)^2}{\sum\limits_{i<j}(X^{ij})^2 \sum\limits_{i<j}(y^{ij})^2 - \left(\sum\limits_{i<j} X^{ij}y^{ij}\right)^2}$$

In particular, for $SO(3)$, for example, $K_\sigma = \frac{1}{4}$.

We introduce a basis on the tangent space of the $T_E SO(3)$ as follows

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This basis is orthonormal and the Lie bracket of the basis elements is defined by the table

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

If we compare the basis vectors $\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$ to the basis X_1, X_2, X_3 on R^3, then the linear isomorphism between $T_E SO(3)$ and R^3 is an isometry.

Consider the three-dimensional sphere S^3 of radius 2 with the standard Riemannian metric. The vectors X_1, X_2, X_3 in the standard metric of the sphere S^3 form an orthonormal basis in $T_e S^3$, and as left-invariant fields - an orthonormal basis for any

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point S^3. The scalar product $< X, Y > |_p = \lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3$, for any $X, Y \in T_eS^3$, such that

$$X = \lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3,\ Y = \mu_1X_1 + \mu_2X_2 + \mu_3X_3.$$

We introduce a new metric $\ll \cdot, \cdot \gg$ on S^3 as follows

$$\ll X, Y \gg = \frac{1}{3}\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3.$$

The difference between the metric $\ll \cdot, \cdot \gg$ and the standard metric is that each tangent space "contracts" in the direction of the vector X_1.

Consider the Lie group $G = S^3 \times R$ with the metric of the direct product of sphere and line. This metric is bi-invariant. The Lie algebra of the group G is obtained from the Lie algebra of the group S^3 by adding X_1, X_2, X_3 tangent to R of the unit vector X_4 for which $[X_i, X_4] = 0$ holds for all i. The resulting basis is orthonormal.

Theorem 5. *The manifold $G = S^3 \times R$ is a variety of strictly positive bounded sectional curvature.*

Proof. Consider the vector fields on the group G

$$Z_1 = \frac{1}{\sqrt{3}}X_1 + \frac{\sqrt{2}}{\sqrt{3}}X_4,\ \ Z_2 = -\frac{\sqrt{2}}{\sqrt{3}}X_1 + \frac{1}{\sqrt{3}}X_4.$$

Let $\pi : G \to S^3 \times \{0\}$ be a projection along the integral curves of the vector field Z_2. The sphere $S^3 \times \{0\}$ with the metric $\ll \cdot, \cdot \gg$ is denoted by M. Moreover, the map $\pi : G \to M$ is a Riemannian submersion, where the field Z_1 is horizontal, and the field Z_2 is vertical, so that

$$|d\pi Z_1| = |Z_1| = 1,\ \ |d\pi Z_2| = 0.$$

Since $X_1 = \frac{1}{\sqrt{3}}Z_1 - \sqrt{\frac{2}{3}}Z_2$, then $|d\pi X_1| = \frac{1}{\sqrt{3}}$.

Using the O'Neil's formula, we calculate the sectional curvature K_σ of the manifold M. Suppose that $X, Y \in T_pM$ is

$$X = \sqrt{3}\lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3,\ Y = \sqrt{3}\mu_1X_1 + mu_2X_2 + \mu_3X_3,$$

so that

$$|X \wedge Y|^2_M = \ll X, X \gg \ll Y, Y \gg - \ll X, Y \gg^2 = \nu_1^2 + \nu_2^2 + \nu_3^2,$$

where $\nu_1 = \lambda_2\mu_3 - \lambda_3\mu_2,\quad \nu_2 = \lambda_3\mu_1 - \lambda_1\mu_3,\quad \nu_3 = \lambda_1\mu_2 - \lambda_2\mu_1$.

Horizontal lifts of vector fields X, Y in G are vector fields

$$\overline{X} = \lambda_1Z_1 + \lambda_2X_2 + \lambda_3X_3 = \tfrac{1}{\sqrt{3}}\lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \sqrt{\tfrac{2}{3}}\lambda_1X_4,$$
$$\overline{Y} = \mu_1Z_1 + \lambda_2X_2 + \lambda_3X_3 = \tfrac{1}{\sqrt{3}}\mu_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \sqrt{\tfrac{2}{3}}\mu_1X_4.$$

Their Lie bracket is

$$\left[\overline{X}, \overline{Y}\right] = \nu_1 X_1 + \frac{1}{\sqrt{3}} \nu_2 X_2 + \frac{1}{\sqrt{3}} \nu_3 X_3$$

and the vertical component is

$$\left[\overline{X}, \overline{Y}\right]^V = -\sqrt{\frac{2}{3}} \nu_1 Z_2.$$

The sectional curvature K_σ of the manifold M for $\sigma = X \wedge Y$ is equal to

$$K_\sigma = \frac{k(X,Y)}{|X \wedge Y|_M^2} = \frac{k(\overline{X}, \overline{Y})_G + \frac{3}{4}(\left[\overline{X}, \overline{Y}\right]^V)_G^2}{|X \wedge Y|_M^2} =$$

$$= (v_1^2 + v_2^2 + v_3^2)^{-1} \left(\frac{1}{4} \left[\overline{X}, \overline{Y}\right]_G^2 + \frac{3}{4} \left(\left[\overline{X}, \overline{Y}\right]^V \right)_G^2 \right) =$$

$$= \frac{\left(1 + 3\left(\sqrt{\frac{2}{3}}\right)^2\right) v_1^2 + \left(\frac{1}{\sqrt{3}}\right)^2 (\nu_2^2 + \nu_3^2)}{4\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right)} = \frac{9\nu_1^2 + (\nu_2^2 + \nu_3^2)}{12\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right)}.$$

Thus, the sectional curvature of the manifold M is strictly positive and satisfies the inequality $\frac{1}{12} \le K_\sigma \le \frac{3}{4}$. The theorem is proved. $\square$

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